HARNACK ESTIMATE FOR MEAN CURVATURE FLOW ON THE SPHERE

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ABSTRACT. We consider the evolution of hypersurfaces on the unit sphere $\mathbb{S}^{n+1}$ by their mean curvature. We prove a differential Harnack inequality for any weakly convex solution to the mean curvature flow. As an application, by applying an Aleksandrov reflection argument, we classify convex, ancient solutions of the mean curvature flow on the sphere.

We study convex hypersurfaces M_0, $n \geq 2$, without boundary, which are smoothly embedded in $\mathbb{S}^{n+1}$. Let $F_0 : M^n \to \mathbb{S}^{n+1}$ be a smooth embedding of M_0. The mean curvature flow with initial data F_0 is a family of hypersurfaces given by embeddings $F : M^n \times [0, T) \to \mathbb{S}^{n+1}$, $F(\cdot, 0) = F_0(\cdot)$ that moves in the direction of the unit normal vector ν with speed equal to the mean curvature H, the trace of the second fundamental form $A(V, V)$ over the tangent vectors V, e.g.,

$$\partial_t F(\cdot, t) = -H(\cdot, t)\nu(\cdot, t).$$

Here, $H(p, t)$ is the mean curvature of the hypersurface $M_t := F(M^n, t)$ at the point $F(p, t)$ where the unit outer normal vector is $\nu(p, t)$. Let D denote the Levi-Civita connection of $\mathbb{S}^{n+1}$.

Theorem (Harnack estimate). *Suppose M_t is a smooth, weakly convex solution of the mean curvature flow on the time interval $[0, T)$. For $t > 0$ we have*

$$\partial_t H + \frac{H}{2t} + 2D_V H + A(V, V) - nH \geq 0$$

for all tangent vectors V.

Comparing this with Hamilton's Harnack inequality [5] for the mean curvature flow in $\mathbb{R}^{n+1}$, here we gained a bonus term $-nH$ through the ambient space $\mathbb{S}^{n+1}$. This allows us, by employing a parabolic Aleksandrov reflection argument first developed in [6, 7, 8], a rather convenient classification of ancient, smooth, weakly convex solutions of the mean curvature flow.

Theorem (Classification of ancient solutions). *The only ancient, weakly convex, embedded solutions of the mean curvature flow on the sphere are equators and shrinking geodesic spheres.*

Let us note that such a classification has been obtained already in [13, Theorem 6.1], with a relatively short proof applying the maximum principle to the quantity $(\|A\|^2 - \frac{1}{n}H^2)/H^2$. As usual for such arguments, the Codazzi equation is employed; therefore, the result pertains to $n \geq 2$. The first author and Louie

[4] used a completely different argument to obtain the classification for $n = 1$. First, a Harnack inequality is obtained, which ensures that the backwards limit as $t \to -\infty$ is an equator. Then the Aleksandrov reflection argument is employed to show maximal reflection symmetry, immediately resulting in the classification. Our work here extends the techniques of [4] from $n = 1$ to $n \geq 2$ formally following the same procedure of obtaining the Harnack inequality and then using the Aleksandrov reflection to show the maximal reflection symmetry. We believe, that unlike the methods in [13], our argument is more broadly applicable: once the Harnack inequality is obtained for a curvature flow, the remainder of the argument holds with very little modification. This theme will be explored for a range of curvature flows similar to those studied in [2] in a forthcoming paper. In [2], the Gauss map parametrization and support function are used to great effect, considerably simplifying the computations. A genuine difficulty on the sphere is obtaining a similar useful parametrization under which we can perform the calculations.

Before moving on to the details, a few words about the broader context are in order. It is well known that ancient solutions arise as singularity models for curvature flows (e.g. [12]); therefore, their study is of great importance. On the unit sphere, as opposed to the Euclidean case studied in [13], where classification can be obtained only with additional assumptions such as pinching, we find a complete classification and strong rigidity. The difference is the compactness and positive curvature of the sphere. We might conjecture that for the mean curvature flow in a positively curved, compact background, there exists at most one non-trivial ancient solution emanating from each closed minimal hypersurface. However, in general, it may be too much to hope that such ancient solutions exist.

Classifications have been obtained for the curve shortening flow in the plane [9] and the Ricci flow of surfaces [10]. Both examples include self-similar shrinkers (round circles for the former and constant curvature spheres for the latter) and a unique Type II ancient solution with curvature blowing up faster than $|t|$ as $t \to -\infty$ (the Angenent oval [paper clip] in the former case and the Rosenau in the latter). Our results here rule out the possibility of such Type II singularities and show that *in any dimension*, the only weakly convex, ancient solutions are the "self-similar" shrinkers: here, self-similar means conformally equivalent to the self-similar family of shrinking spheres in Euclidean space after realizing the sphere as (locally) conformally equivalent to Euclidean space.

In higher dimensions, classification results for ancient solutions in Euclidean space are less complete and a wider range of possibilities may occur. As mentioned above, the results of [13] require additional pinching assumptions to deduce that such ancient solutions are shrinking spheres. Wang [18] studied translating ancient solutions and discovered the existence of non-rotationally symmetric translators (though they blow down to spheres or cylinders). Furthermore, there are solutions asymptotic to "ovals" in that the center looks like a cylinder, but the ends look like "bowls" [1, 14, 19]. These solutions have similarities to the Angenent oval in the plane.

In the sphere, we may consider an ancient solution moving by a rotation to be a translating solution, but we quickly realize that no such convex solutions can exist because convexity and the containment principle force the flow to lie in a fixed hemisphere at all times. Our results also show that no non-self-similar ancient solutions exist. The phenomena of becoming more and more oval as $t \to -\infty$ for

convex solutions appears to be ruled out by the compactness of the sphere; such a solution would have to touch an equator at the "ends" and "fatten" out in the middle, which forces it to touch the equator in the middle and thus everywhere (by convexity).

Lastly, the connection between Harnack inequalities, ancient solutions and solitons deserves mention. In Euclidean space, the classic work [5] espouses the philosophy that Harnack inequality should be an equality on solitons. We have already observed that unlike Euclidean space, no translating convex solitons exist in the sphere. The equivalent of homothetic solitons may be taken to be those solutions of the mean curvature flow following the integral curves of the conformal "position" vector field $\sin(d)\nabla d$, where d denotes the distance to a fixed point (the point of collapse). This vector field generates a one-parameter subgroup of conformal transformations, under which the mean curvature flow is not invariant, though the family of shrinking geodesic spheres is such a soliton and hence these conformal solitons are of great interest. The lack of conformal invariance suggests that conformal solitons do not satisfy equality in the Harnack inequality, and this is indeed the case for the shrinking spheres. Interested readers should consult [15, 16, 17] for more details on solitons interpreted as the flow along integral curves of a vector field, particularly [3] for details on conformal solitons.

Acknowledgement

1. Preliminaries

Let $g = \{g_{ij}\}$, $A = \{h_{ij}\}$, and Rm_{ijkl} denote, in order, the induced metric, the second fundamental form, and the curvature tensor of M^n. The mean curvature of M^n is the trace of the second fundamental form with respect to g, $H = g^{ij}h_{ij}$. Let $\overline{g}$ and $\overline{Rm}_{\alpha\beta\gamma\theta}$ denote, respectively, the metric and the curvature tensor of $\mathbb{S}^{n+1}$. Greek indices run through $\{0, \cdots, n\}$ and Latin indices belong to the set $\{1, \cdots, n\}$.

Write ν for the outer unit normal to M_t. For a fixed time, we choose a local orthonormal frame $\{\partial_0 = \nu, \cdots, \partial_i = \frac{\partial F(\cdot, t)}{\partial x_i}, \cdots, \partial_n\}$ in $\mathbb{S}^{n+1}$. We use the following standard notation

$$h_i^j = g^{mj}h_{im}$$
$$(h^2)_i^j = g^{mj}g^{rs}h_{ir}h_{sm}$$
$$|A|^2 = g^{ij}g^{kl}h_{ik}h_{lj} = h_{ij}h^{ij}$$
$$C = g^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj} = h_i^k h_k^l h_l^i.$$

Here, $\{g^{ij}\}$ is the inverse matrix of $\{g_{ij}\}$.

The relations between A, Rm, and $\overline{Rm}$ are given by the Gauß and Codazzi equations:

$$Rm_{ijkl} = \overline{Rm}_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}$$
$$\nabla_i h_{jk} = \nabla_k h_{ij}.$$

Moreover, ∇_i and Δ commute as follows

$$(\nabla_i \Delta - \Delta \nabla_i)f = -Rc_i^j \nabla_j f$$

for all smooth functions on M^n. Therefore, in view of $\bar{R}_{\alpha\beta\gamma\theta} = \lambda(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\theta} - \bar{g}_{\alpha\theta}\bar{g}_{\beta\gamma})$, $\lambda = 1$, and the Gauß equation we have

$$(1.1) \qquad (\nabla_i \Delta - \Delta \nabla_i)H = ((h^2)_i^m - Hh_i^m - (n-1)\delta_i^m \lambda)\nabla_m H.$$

2. EVOLUTION EQUATIONS

In this section, we assume that M_t is a strictly convex solution of the mean curvature flow.

Lemma 1. *The following evolution equations hold under the mean curvature flow:*

(1) $\partial_t g_{ij} = -2Hh_{ij}$
(2) $\partial_t g^{ij} = 2Hg^{im}g^{jn}h_{ij} = 2Hh^{ij}$
(3) $\partial_t h_i^j = \nabla^j \nabla_i H + H(h^2)_i^j + \lambda H \delta_i^j$
(4) $\partial_t h_i^j = \Delta h_i^j + |A|^2 h_i^j + \lambda\{2H\delta_i^j - nh_i^j\}$
(5) $\partial_t h_{ij} = \Delta h_{ij} + |A|^2 h_{ij} - 2H(h^2)_{ij} + \lambda\{2Hg_{ij} - nh_{ij}\}$
(6) $\partial_t H = \Delta H + H|A|^2 + n\lambda H$
(7) $(\partial_t \Delta - \Delta \partial_t)H = 2Hh^{ij}\nabla_i \nabla_j H + 2h^{ij}\nabla_i H \nabla_j H.$

Proof. The computations for $(1-6)$ are straightforward (see for example [11]). To obtain (7), we note that

$$(\partial_t \Delta - \Delta \partial_t)H = \left(\partial_i \partial_j H - \Gamma_{ij}^k \partial_k H\right)\partial_t g^{ij} - g^{ij}\partial_k H \partial_t \Gamma_{ij}^k,$$

where $\Gamma_{ij}^k = g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$. Since $\partial_t \Gamma_{ij}^k$ is tensorial, using (1) and (2) we may carry out our calculations in a normal frame to prove the claim. $\square$

Lemma 2. *Under the mean curvature flow we have*

$$\partial_t(\partial_t H) = \Delta \partial_t H + 4Hh^{ij}\nabla_i \nabla_j H + 2h^{ij}\nabla_i H \nabla_j H$$
$$+ (|A|^2 + n\lambda)(\partial_t H) + 2H^2 C + 2\lambda H^3,$$

and

$$\nabla_i \partial_t H = \Delta \nabla_i H + \nabla_i(|A|^2 H) + ((h^2)_i^m - Hh_i^m)\nabla_m H + \lambda \nabla_i H.$$

Proof. Using $(3),(6)$ and (7) in Lemma 2, we calculate

$$\begin{aligned}
\partial_t(\partial_t H) &= \partial_t(\Delta H + H|A|^2 + n\lambda H) \\
&= \Delta \partial_t H + 2Hh^{ij}\nabla_i \nabla_j H + 2h^{ij}\nabla_i H \nabla_j H \\
&\quad + (|A|^2 + n\lambda)(\partial_t H) + H(\partial_t|A|^2) \\
&= \Delta \partial_t H + 2Hh^{ij}\nabla_i \nabla_j H + 2h^{ij}\nabla_i H \nabla_j H \\
&\quad + (|A|^2 + n\lambda)(\partial_t H) + H(2h^{ij}\nabla_i \nabla_j H + 2HC + 2\lambda H^2) \\
&= \Delta \partial_t H + 4Hh^{ij}\nabla_i \nabla_j H + 2h^{ij}\nabla_i H \nabla_j H \\
&\quad + (|A|^2 + n\lambda)(\partial_t H) + 2H^2 C + 2\lambda H^3.
\end{aligned}$$

To obtain the second evolution equation, we use identity (1.1) and part (6) of Lemma 2:

$$\begin{aligned}
\nabla_i \partial_t H &= \nabla_i(\Delta H + |A|^2 H + n\lambda H) \\
&= \Delta \nabla_i H + \nabla_i(|A|^2 H) + n\lambda \nabla_i H \\
&\quad + ((h^2)_i^m - H h_i^m - (n-1)\delta_i^m \lambda)\nabla_m H \\
&= \Delta \nabla_i H + \nabla_i(|A|^2 H) + ((h^2)_i^m - H h_i^m)\nabla_m H + \lambda \nabla_i H.
\end{aligned}$$

$\square$

In the sequel, $\{b^{ij}\}$ denotes the inverse of the second fundamental form.

Lemma 3.

$$\begin{aligned}
\partial_t(b^{ij}\nabla_i H \nabla_j H) \leq &- b^{im}b^{jn}\Delta h_{mn}\nabla_i H \nabla_j H + 2b^{ij}\nabla_j H \Delta \nabla_i H \\
&+ 2h^{ij}\nabla_i H \nabla_j H + |A|^2 b^{ij}\nabla_i H \nabla_j H + 2H b^{ij}\nabla_i |A|^2 \nabla_j H \\
&+ n\lambda b^{ij}\nabla_i H \nabla_j H.
\end{aligned}$$

Proof. Observe

$$-\lambda b^{im}b^{jn}\{2H g_{mn} - n h_{mn}\}\nabla_i H \nabla_j H + 2\lambda b^{ij}\nabla_i H \nabla_j H \leq n\lambda b^{ij}\nabla_i H \nabla_j H$$

and

$$\begin{aligned}
\partial_t b^{ij} &= -b^{im}b^{jn}\partial_t h_{mn} \\
&= -b^{im}b^{jn}(\Delta h_{mn} + |A|^2 h_{mn} - 2H(h^2)_{mn}) \\
&\quad - \lambda b^{im}b^{jn}\{2H g_{mn} - n h_{mn}\} \\
&= -b^{im}b^{jn}\Delta h_{mn} - |A|^2 b^{ij} + 2H g^{ij} \\
&\quad - \lambda b^{im}b^{jn}\{2H g_{mn} - n h_{mn}\}.
\end{aligned}$$

Therefore, using the second part of Lemma 2, we obtain

$$\begin{aligned}
\partial_t(b^{ij}\nabla_i H \nabla_j H) = &- b^{im}b^{jn}(\Delta h_{mn} + |A|^2 h_{mn} - 2H(h^2)_{mn})\nabla_i H \nabla_j H \\
&- \lambda b^{im}b^{jn}\{2H g_{mn} - n h_{mn}\}\nabla_i H \nabla_j H \\
&+ 2b^{ij}(\Delta \nabla_i H + \nabla_i(|A|^2 H) + ((h^2)_i^m - H h_i^m)\nabla_m H + \lambda \nabla_i H)\nabla_j H \\
\leq &- b^{im}b^{jn}\Delta h_{mn}\nabla_i H \nabla_j H + 2b^{ij}\nabla_j H \Delta \nabla_i H \\
&+ 2b^{im}b^{jn}H(h^2)_{mn}\nabla_i H \nabla_j H \\
&+ |A|^2 b^{ij}\nabla_i H \nabla_j H + 2H b^{ij}\nabla_i |A|^2 \nabla_j H \\
&+ 2b^{ij}((h^2)_i^m - H h_i^m)\nabla_m H \nabla_j H \\
&+ n\lambda b^{ij}\nabla_i H \nabla_j H \\
= &- b^{im}b^{jn}\Delta h_{mn}\nabla_i H \nabla_j H + 2b^{ij}\nabla_j H \Delta \nabla_i H \\
&+ 2h^{ij}\nabla_i H \nabla_j H + |A|^2 b^{ij}\nabla_i H \nabla_j H + 2H b^{ij}\nabla_i |A|^2 \nabla_j H \\
&+ n\lambda b^{ij}\nabla_i H \nabla_j H.
\end{aligned}$$

$\square$

Using the identities

$$\nabla_m b^{ij} = -b^{ik}b^{jl}\nabla_m h_{kl}$$

and

$$\Delta b^{ij} = - b^{im}b^{jn}\Delta h_{mn} + \{b^{ir}b^{ks}b^{jl} + b^{ik}b^{jr}b^{ls}\}g^{pq}\nabla_p h_{rs}\nabla_q h_{kl},$$

we compute

$$\Delta(b^{ij}\nabla_i H\nabla_j H) = - b^{im}b^{jn}\nabla_i H\nabla_j H\Delta h_{mn} + 2b^{ij}\nabla_j H\Delta\nabla_i H$$
$$+ 2\{b^{ir}b^{ks}b^{jl}\}\nabla_i H\nabla_j Hg^{pq}\nabla_p h_{rs}\nabla_q h_{kl}$$
$$- 4g^{mn}b^{ik}b^{jl}\nabla_m h_{kl}\nabla_n\nabla_i H\nabla_j H + 2b^{ij}g^{mn}(\nabla_m\nabla_i H\nabla_n\nabla_j H).$$

Thus we have proved:

Lemma 4.

$$\partial_t(b^{ij}\nabla_i H\nabla_j H) \leq \Delta(b^{ij}\nabla_i H\nabla_j H)$$
$$- 2\{b^{ir}b^{ks}b^{jl}\}\nabla_i H\nabla_j Hg^{pq}\nabla_p h_{rs}\nabla_q h_{kl}$$
$$+ 4g^{mn}b^{ik}b^{jl}\nabla_m h_{kl}\nabla_n\nabla_i H\nabla_j H - 2b^{ij}g^{mn}(\nabla_m\nabla_i H\nabla_n\nabla_j H)$$
$$+ 2h^{ij}\nabla_i H\nabla_j H + |A|^2 b^{ij}\nabla_i H\nabla_j H + 2Hb^{ij}\nabla_i|A|^2\nabla_j H$$
$$+ n\lambda b^{ij}\nabla_i H\nabla_j H.$$

Lemma 5. *Define* $\Theta := \partial_t H - b^{ij}\nabla_i H\nabla_j H$. *Then*

$$\partial_t\Theta \geq \Delta\Theta + \frac{2(\Theta - n\lambda H)^2}{H} + (|A|^2 + n\lambda)\Theta$$
$$+ 2\{g^{mq}b^{np} - \frac{g^{mn}g^{pq}}{H}\}\eta_{mn}\eta_{pq} + 2\lambda H^3,$$

where

$$\eta_{mn} = \nabla_m\nabla_n H + H(h^2)_{mn} - b^{rs}\nabla_r H\nabla_s h_{mn}.$$

Proof. Note that

$$2\frac{(\Theta - n\lambda H)^2}{H} = 2\frac{(\Delta H)^2}{H} + 4|A|^2\Delta H - 4\frac{\Delta H}{H}b^{ij}\nabla_i H\nabla_j H$$
$$+ 2|A|^4 H - 4|A|^2 b^{ij}\nabla_i H\nabla_j H$$
$$+ \frac{2}{H}\left(b^{ij}\nabla_i H\nabla_j H\right)^2$$

and

$$2\{g^{mq}b^{np} - \frac{g^{mn}g^{pq}}{H}\}\eta_{mn}\eta_{pq}$$
$$= 2g^{mq}b^{np}\nabla_m\nabla_n H\nabla_p\nabla_q H - 2\frac{(\Delta H)^2}{H}$$
$$+ 4Hh^{ij}\nabla_i\nabla_j H - 4|A|^2\Delta H$$
$$- 4g^{mq}b^{np}b^{rs}\nabla_m\nabla_n H\nabla_r H\nabla_s h_{pq} + 4\frac{\Delta H}{H}b^{ij}\nabla_i H\nabla_j H$$
$$+ 2H^2 C - 2|A|^4 H$$
$$- 2Hb^{ij}\nabla_i H\nabla_j|A|^2 + 4|A|^2 b^{ij}\nabla_i H\nabla_j H$$
$$+ 2\{b^{ir}b^{ks}b^{jl}\}\nabla_i H\nabla_j Hg^{pq}\nabla_p h_{rs}\nabla_q h_{kl} - \frac{2}{H}\left(b^{ij}\nabla_i H\nabla_j H\right)^2.$$

Now the claim follows from adding up these last two identities and considering Lemmas 2 and 4. $\qquad\square$

3. Harnack estimate and Backwards Convergence

3.1. **Harnack estimate.** If M_0 is not an equator, the strong parabolic maximum principle and the evolution equation of h_i^j in Lemma 2 imply that for any $t > 0$, M_t is strictly convex. For strictly convex hypersurfaces,

$$A(V, V) + 2D_V H$$

is minimized by $V = (V^1, \cdots, V^i = -b^{ij}\nabla_i H, \cdots, V^n)$ and thus to prove the main theorem, it suffices to verify that for all $t > 0$

$$\partial_t H - b^{ij}\nabla_i H \nabla_j H - n\lambda H + \frac{H}{2t} \geq 0.$$

Fix $0 < \varepsilon < T$. We will apply the maximum principle to

$$Q := \frac{\Theta - n\lambda H}{H} = \frac{\Delta H + |A|^2 H - b^{ij}\nabla_i H \nabla_j H}{H}$$

on the time interval $[\varepsilon, T)$. Using Lemmas 1, 5, and the inverse-concavity of the mean curvature, we calculate

$$\partial_t Q = \partial_t \frac{\Theta}{H} \geq \Delta Q + 2\langle \nabla Q, \frac{\nabla H}{H}\rangle + 2Q^2 + 2\lambda H^2$$
$$+ 2\frac{\{g^{mq}b^{np} - \frac{g^{mn}g^{pq}}{H}\}\eta_{mn}\eta_{pq}}{H}$$
$$\geq \Delta Q + 2\langle \nabla Q, \frac{\nabla H}{H}\rangle + 2Q^2.$$

An ODE comparison with $q(t) = -\frac{1}{2(t-\varepsilon)}$ which satisfies $\frac{d}{dt}q(t) = 2q^2(t)$ and $\lim_{t\to\varepsilon^+} q(t) = -\infty$ shows that $Q(\cdot, t) \geq q(t)$ for any $t > \varepsilon$. Allowing $\varepsilon \to 0$ completes the proof of the Harnack estimate.

3.2. **Backwards convergence.**

Lemma 6. *Any weakly convex ancient solution of the mean curvature flow satisfies*

$$|A| \leq c_0 \exp(nt)$$

for $t \leq 0$ for some c_0 depending only on M_0.

Proof. By the strong parabolic maximum principle, any weakly convex ancient solution of the mean curvature flow must become strictly convex, unless it is a non-moving equator. By Theorem 11 for strictly convex, ancient solutions we have

$$\frac{\partial_t H - b^{ij}\nabla_i H \nabla_j H - n\lambda H}{H} \geq 0.$$

Since $\lambda = 1$, we get

$$\partial_t \log H \geq n.$$

Integrating both sides of this inequality against dt on the time interval $[t, 0]$ gives

$$H(\cdot, t) \leq H(\cdot, 0) \exp(nt).$$

$\qquad\square$

Lemma 7. *Any weakly convex ancient solution of the mean curvature flow satisfies*

$$|\nabla^m A|^2 \leq c_m \exp(2nt)$$

for $t \leq 0$, where c_m depends only on M_0 and m.

Proof. The proof follows by induction on m. Using the fourth evolution equation in Lemma 1, $\partial_t \Gamma_{ij}^k = A * \nabla A$ and that the commutator $[\nabla^k, \Delta]T$ is given by

$$[\nabla^k, \Delta]T = \sum_{j=0}^{k} \nabla^j Rm * \nabla^{k-j}T$$

for any tensor T, we can compute the following evolution equations:

(1)
$$\partial_t |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4 + 2\lambda(2H^2 - n|A|^2)$$

(2)
$$\begin{aligned}
\partial_t |\nabla^m A|^2 =& \Delta |\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 \\
&+ \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A \\
&+ \lambda \nabla^m A * \nabla^m A.
\end{aligned}$$

On the other hand, by Lemma 6 we get

$$\partial_t |A|^2 \leq \Delta |A|^2 - 2|\nabla A|^2 + c_1 \exp(2nt),$$

and

$$\partial_t |\nabla A|^2 \leq \Delta |\nabla A|^2 + c_2 |\nabla A|^2 + c_3 \exp(2nt).$$

Here, $c_1, c_2, c_3 \geq 0$ are independent of t. Therefore,

$$\begin{aligned}
\partial_t ((t-s)|\nabla A|^2 + b|A|^2) \leq& \Delta((t-s)|\nabla A|^2 + b|A|^2) \\
&+ (1 - 2b + (t-s)c_2)|\nabla A|^2 + (c_1 b + c_3)\exp(2nt).
\end{aligned}$$

We want to apply the maximum principle on the time interval $[s, s+1]$. We choose b large enough, independent of t, so that the coefficient of $|\nabla A|^2$ becomes negative. Thus the maximum principle implies that

$$(t-s)|\nabla A|^2 \leq c_4 \exp(2nt)(1 - \exp(2n(s-t))) + bc_0^2 \exp(2ns).$$

In particular for $t = s + 1$ we have

$$|\nabla A|^2(\cdot, t) \leq c_5 \exp(2nt).$$

This verifies the bound on $|\nabla A|$. Higher derivative estimates follow by induction and using the test function $(t-s)|\nabla^m A|^2 + b_{m-1}|\nabla^{m-1} A|^2$:

$$\partial_t |\nabla^{m-1} A|^2 \leq \Delta |\nabla^{m-1} A|^2 - 2|\nabla^m A|^2 + c_1 \exp(2nt).$$
$$\partial_t |\nabla^m A|^2 \leq \Delta |\nabla^m A|^2 + c_2 |\nabla^m A|^2 + c_3 \exp(2nt).$$

$\square$

Having established the higher derivative curvature bounds, convexity of M_t implies that the backwards limit of M_t is independent of subsequences. Therefore we have proved:

Theorem 8. *Let M_t be an ancient, embedded, weakly convex solution of the mean curvature flow. Then M_t converges exponentially fast in C^∞ to a unique equator $M_{-\infty}$ as $t \to -\infty$.*

4. CLASSIFICATION OF ANCIENT SOLUTIONS

In this section, we use Theorem 8 to classify convex, embedded ancient solution of the mean curvature flow on $\mathbb{S}^{n+1}$. The proof uses the Aleksandrov reflection as in [4].

We begin with some preliminaries of the Aleksandrov reflection on $\mathbb{S}^{n+1}$. First, we will work relative to the limiting equator obtained in Theorem 8. Let E be an equator that bounds the *open* hemispheres $H^\pm$ with centers $\pm \mathrm{p}_0$, and let $e = \overrightarrow{\mathrm{O}\,\mathrm{p}_0}$ be the unit vector in $\mathbb{R}^{n+2}$ that points from the origin O to p_0. Let $\rho(x) = d_{\mathbb{S}^{n+1}}(\mathrm{p}_0, x)$ denote the spherical distance from p_0 to $x \in \mathbb{S}^{n+1}$. The radial projection onto E is the map $x \in \mathbb{S}^{n+1} \mapsto \pi(x) \in E$, where π is the nearest point on E to x. If $x \neq \pm \mathrm{p}_0$, then $\pi(x)$ is a single point. If $x = \pm \mathrm{p}_0$, then $\pi(x) = E$. In any event, given $y \in \pi(x)$, there is a unique length minimizing geodesic joining x to y and this geodesic must pass through $\pm \mathrm{p}_0$.

It is convenient to make use of the ambient $\mathbb{R}^{n+2}$ and define the height function $\mathrm{h}(x) = \langle x, e \rangle$. The radial distance is related to the height function via

$$\mathrm{h}(x) = \cos(\rho(x))$$

which is monotonically decreasing in ρ.

Now for the Aleksandrov reflection, let $\tilde{\mathrm{V}} \in \mathbb{R}^{n+2}$ be any unit vector that $\langle \tilde{\mathrm{V}}, e \rangle < 0$. Let $P_{\tilde{\mathrm{V}}} = \tilde{\mathrm{V}}^\perp$ be the hyperplane through the origin with the normal vector $\tilde{\mathrm{V}}$. Let $H_{\tilde{\mathrm{V}}}^\pm = \{\pm \langle x, \tilde{\mathrm{V}} \rangle > 0\}$ denote the halfspaces with the boundary $P_{\tilde{\mathrm{V}}}$. For any subset $S \subset \mathbb{S}^{n+1}$, write $S_{\tilde{\mathrm{V}}}^\pm = S \cap H_{\tilde{\mathrm{V}}}^\pm$. Lastly, let $\delta > 0$ denote the angle $\tilde{\mathrm{V}}$ makes with E; therefore, $\sin \delta = \langle \tilde{\mathrm{V}}, -e \rangle$.

Definition 9. The Aleksandrov reflection across $P_{\tilde{\mathrm{V}}}$ is the map defined by

$$R_{\tilde{\mathrm{V}}} : x \in \mathbb{R}^{n+1} \mapsto x - 2\langle x, \tilde{\mathrm{V}} \rangle \tilde{\mathrm{V}}.$$

This map is an (orientation reversing) isometry of $\mathbb{R}^{n+2}$ fixing $P_{\tilde{\mathrm{V}}}$ and in particular fixing the origin. Therefore, it induces an isometry of $\mathbb{S}^{n+1}$. For $x \in E$, we have $\langle x, e \rangle = 0$ and

$$\mathrm{h}(R_{\tilde{\mathrm{V}}}(x)) = \langle e, x - 2\langle x, \tilde{\mathrm{V}} \rangle \tilde{\mathrm{V}} \rangle = 2 \sin \delta \langle x, \tilde{\mathrm{V}} \rangle.$$

In the case $x \in E^+$, $\mathrm{h}(R_{\tilde{\mathrm{V}}}(x)) > 0$, and in the case $x \in E \cap P_{\tilde{\mathrm{V}}}$, $\mathrm{h}(R_{\tilde{\mathrm{V}}}(x)) = 0$.

In geodesic polar coordinates, $(\rho, \sigma) \in (0, \pi) \times E \simeq \mathbb{S}^{n+1} \setminus \{\pm \mathrm{p}_0\}$, a smooth, closed hypersurface $M \subset H_{\tilde{\mathrm{V}}}^+$ that bounds a region in $H_{\tilde{\mathrm{V}}}$ is a smooth graph $(f(\sigma), \sigma)$ over E if and only if its (outer) normal ν satisfies $\langle \nu, e \rangle < 0$ (e.q., M has no vertical tangents). In particular, for all $\epsilon > 0$ there is a $\xi > 0$ such that if M is a graph with $\langle \nu, e \rangle < -\xi$ and N is ϵ-close to M in C^∞, then N is a graph over E.

Theorem 10. *Let M_t be a convex, embedded ancient solution of the mean curvature flow on $\mathbb{S}^{n+1}$. Then M_t is a family of shrinking geodesic spheres emanating from an equator at $t = -\infty$.*

Proof. Let $E = M_{-\infty} \simeq \mathbb{S}^n$ be the limiting equator $\lim\limits_{t \to -\infty} M_t$.

Since M_t smoothly converges to E as $t \to -\infty$, we may write M_t as the graph of a smooth positive function over E in the geodesic polar coordinates, $M_t = \{(f_t(\sigma), \sigma) \in (0, \pi) \times E\}$. We have $f_t \to \pi/2$ smoothly and uniformly in C^k, for any k, as $t \to -\infty$. Moreover, for $\delta \in (0, \pi/4)$, $R_{\tilde{\mathrm{V}}}(E) = \{(g_{-\infty}(\sigma), \sigma)\}$ is a graph over

E. Note that $R_{\tilde{V}}(E)$ is not a graph when $\delta = \pi/4$ (it is an equator perpendicular to E), so let us fix a $\delta_0 \in (0, \pi/4)$ to give us a little room. Then it follows from [4, Lemma 5.2] that both $M_{t\tilde{V}}^-$ and $R_{\tilde{V}}((M_t)_{\tilde{V}}^+)$ are non-empty and are graphs for all times $t \in (-\infty, S)$ with $S > -\infty$ independent of $\delta \in (0, \delta_0)$. Let us write g_t for the graph of $R_{\tilde{V}}(M_t)$ over E.

As noted above, $h(R_{\tilde{V}}(x)) > 0$ on E^+ and $h(x) = \cos(\rho(x))$. Thus, continuity implies that for $\epsilon > 0$ there exits an $\eta > 0$ such that $\rho(R_{\tilde{V}}(x)) < \pi/2 - \epsilon$ provided $x \in E_\eta = \{x \in E : d(x, E \cap P_{\tilde{V}}) > \eta\}$. Since $M_t \to_{C^\infty} E$, we can choose $T_\delta < 0$, such that $d(M_t, E) < \epsilon/2$ for all $t < T_\delta$; that is, $\rho(x) > \pi/2 - \epsilon/2$ for all $x \in M_t$. Now for $x \in M_t^+ \cap \pi^{-1} E_\eta$, since $R_{\tilde{V}}$ is an isometry, we have $d(R_{\tilde{V}}(x), R_{\tilde{V}}(\pi(x))) < \epsilon/2$; therefore, $\rho(R_{\tilde{V}}(x)) < \pi/2 - \epsilon/2$. Consequently, away from the strip $\{x \in E : d(x, E \cap P_{\tilde{V}}) \le \eta\}$, we have $\rho(R_{\tilde{V}}((M_t)_{\tilde{V}}^+)) < \pi/2 - \epsilon/2$ and $\rho(M_t^-) > \pi/2 - \epsilon/2$. That is, $R_{\tilde{V}}((M_t)_{\tilde{V}}^+) > (M_t)_{\tilde{V}}^-$ away from the strip.

Now consider the great circle C in E joining $\pi(x)$ to $R_{\tilde{V}}(\pi(x))$ for $x \in M_t^+$. From the formula $\pi(x) = \frac{x'}{|x'|}$ where $x' = x - \langle x, e \rangle e$ is orthogonal projection onto the hyperplane $e^\perp$ and the fact that

$$V = \frac{1}{2\langle x, \tilde{V} \rangle}(x - R_{\tilde{V}}(x)),$$

we see that C lies in the plane spanned by $\tilde{V}' = \tilde{V} - \langle \tilde{V}, e \rangle e$ and $\pi(x)$. Letting y be the closest point to $\pi(x)$ in the intersection $C \cap P_{\tilde{V}}$, we find that in fact C lies in the plane spanned by $\tilde{V}$ and y. In particular, since

$$\langle \tilde{V}, \tilde{V}' \rangle = \cos^2(\delta),$$

the angle C makes with P is independent of the choice of x. This allows us to employ the Backwards Approximate Symmetry Lemma [4, Lemma 5.1] (which applies whenever f_t converges smoothly to C) to conclude that (possibly by decreasing T_δ) over C

$$R_{\tilde{V}}((M_t)_{\tilde{V}}^+) \ge (M_t)_{\tilde{V}}^-$$

on the strip $\{x \in E : d(x, E \cap P_{\tilde{V}}) \le \eta\}$. Here, T_δ depends only on $\|f_t - \pi/2\|_{C^2}$ and δ; therefore, $T_\delta > -\infty$ is independent of C. Thus we find that

(4.1) $$R_{\tilde{V}}((M_t)_{\tilde{V}}^+) \ge (M_t)_{\tilde{V}}^-$$

everywhere for all $t \in (-\infty, T_\delta)$.

Let us now define T_δ so that $(-\infty, T_\delta)$ is the largest interval on which the relation (4.1) holds. Also define $T = \inf_{\delta \in (0, \delta_0)} T_\delta$. We want to show that $T > -\infty$, and hence that the relation (4.1) holds on the non-empty, open interval $(-\infty, T)$. To show $T > -\infty$, we apply the maximum principle: Recall that both $M_{t\tilde{V}}^-$ and $R_{\tilde{V}}((M_t)_{\tilde{V}}^+)$ are non-empty and are graphs for all $t \in (-\infty, S)$ with $S > -\infty$ independent of δ. Since the relation (4.1) is true on $(-\infty, \min\{T_\delta, S\})$, the maximum principle applied in the time interval $[\min\{T_\delta, S\}/2, S)$ ensures that

$$R_{\tilde{V}}((M_t)_{\tilde{V}}^+) \ge (M_t)_{\tilde{V}}^-$$

for all $t \in (-\infty, S)$ and any $\delta \in (0, \delta_0)$. Hence $T \ge S$.

To complete the proof, we use [4, Proposition 5.3] (which applies in any dimension) to conclude that M_t is a geodesic sphere for all $t \in (-\infty, T)$ and thus for all negative times by the uniqueness of solutions. $\square$

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